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Research Article

On the Effectiveness of Measures of Uncertainty of Basic Belief Assignments

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ABSTRACT:

This article examines many existing measures of uncertainty of basic belief assignments proposed in the literature related to the theory of belief functions. Some measures capture only a particular aspect of the uncertainty, others propose a total measure of uncertainty to characterize the information quality of a source of information. We discuss the effectiveness of these measures with respect to four main important desiderata that we consider essential for the definition of a satisfactory measure of uncertainty, i.e. effective entropy of basic belief assignment.

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1. Introduction

In the classical framework of belief functions, a source of evidence expresses its belief on the possible solutions of a given problem defined with respect to a chosen (finite) frame of discernment (FoD) Θ . This belief is usually characterized by a basic belief assignment (BBA), referred also as a belief mass denoted by $m(.)$. One of the major concerns related with belief functions is how to measure/quantify the uncertainty encompassed by a source of evidence and

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inherent to any BBA. This problem is challenging and of crucial importance, because its effective solution would allow to well characterize any BBA, to make fair comparisons of sources of evidence, to compare fusion results in terms of uncertainty reduction, to achieve a BBA complexity reduction by new approximations methods, etc.

In this article, we make a state-of-the-art survey of most of the existing MoUs available in the literature and point out their theoretical drawbacks to warn the reader about their misuses and irrelevances in applications. This work justifies the requirement for better effective MoUs to make a step ahead in the understanding and characterization of uncertainty in the belief functions framework. There exist several survey papers covering different proposals for measures of uncertainty, among them we must cite by chronological order,[1-14] and more recently in the works of Moral-García and Joaquín Abellán [15] and Jousselmen and co-authors.[16] These papers however do not consider the effectiveness of MoU as we propose in this paper.

In the sequel, we suppose the reader is familiar with the classical (i.e. Shannon) information theory,[17-22] and especially with Shannon entropy measure, and with the theory of belief functions introduced by Shafer.[23] Some of these basics are recalled in an appendix for convenience and for recalling the classical notations.

This paper is organized as follows. In section 2 we present and justify the four essential desiderata that an MoU should satisfy in order to be considered as effective. In section 3 we examine many existing MoUs proposed in the literature over 40 years and check if they pass the effectiveness test, or not. For those that pass successfully the test, we examine in detail in section 4 if they are sufficiently well justified for considering them as a serious candidate for effective MoU to be used in applications. Section 5 concludes this survey and gives some perspectives for future research works.

2. Desiderata for an effective MoU

Our analysis of many existing works on Measures of Uncertainty (MoU) of belief functions reveals that most of MoUs suffer from serious problems, and we explain why in the next section. Here we introduce several very essential desiderata that a satisfactory MoU, denoted by $U(m)$, should satisfy. Some of these desiderata have already been identified in the past by some researchers working towards axiomatic approaches of MoUs, for instance by Klir [8] and Abellán.[12,13,15] Here we keep only the four desiderata that we consider as really important and indispensable, and we justify our choice for these desiderata. We also explain why we consider the other desiderata not fundamental, and why we decide to discard them. The four essential and indispensable desiderata we consider for a satisfactory MoU are mathematically expressed as follows

● **Desideratum D1**: (zero min value of $U(m)$)

$$U(m)=0 \tag{1}$$

if the BBA m defined on the power set 2^Θ of the frame of discernment Θ is focused on a singleton, that is if $m(X)=1$ for some X of 2^Θ with $|X|=1$.

Justification of D1: This desideratum is very natural and intuitive because any particular BBA for which $m(X)=1$ with $|X|=1$ characterizes the certainty of a singleton X, which is one of the most specific elements of 2^Θ. There is no uncertainty about the choice of this element X characterized by $m(X)=1$ since this element X (the smallest information granule) does not include other smaller elements in it. So, the measure of uncertainty must be minimal, and it can always be arbitrarily set to zero reflecting well such a non-uncertainty case.

● **Desideratum D2**: (increasing of MoU of vacuous BBA)

$$U(m_v^\Theta)<U(m_v^{\Theta'}) \text{ if } |\Theta|<|\Theta'| \tag{2}$$

where m_v^Θ and $m_v^{\Theta'}$ are the vacuous BBAs defined respectively on the frames of discernment (FoDs) Θ and Θ' of cardinalities $|\Theta|$ and $|\Theta'|$.

Justification of D2: This desideratum stipulates that the measure of uncertainty of a total ignorant source of evidence represented by the vacuous BBA must increase with the cardinality of the frame of discernment. This desideratum makes perfect sense because the total ignorant source of evidence on $\Theta=\{\theta_1,\dots,\theta_N\}$ for which $m_v^\Theta(\Theta)=1$ means that one knows absolutely nothing about only N elements, whereas the total ignorant source of evidence on $\Theta'=\{\theta_1,\dots,\theta_N,\theta_{N+1},\dots,\theta_{N'}\}$ for which $m_v^{\Theta'}(\Theta')=1$ means that one knows absolutely nothing about more elements because $N' > N$. This clearly indicates that $m_v^{\Theta'}$ must be considered in fact as more ignorant than m_v^Θ, and the condition (2) reflects this necessity.

● **Desideratum D3**: (compatibility with Shannon entropy)

$$U(m) = -\sum_{X \in \Theta} m(X)\log(m(X)) \tag{3}$$

if the BBA $m(.)$ is a Bayesian BBA defined on the FoD Θ. We recall that any Bayesian BBA commits zero belief mass for all elements of the power set of Θ having their cardinality greater than one.[23]

Justification of D3: This desideratum D3 seems also very natural because Shannon entropy is the most well-known (and justified [20,24-27]) measure used so far to quantify the uncertainty (i.e. the randomness, or variability, also called conflict by some authors) of a probability mass function (pmf). Because any Bayesian BBA induces belief and plausibility functions that coincide with a probability measure, one must have a total coherence of $U(m)$ with Shannon entropy when the BBA is Bayesian if one admits, as we do here, that Shannon entropy is an effective measure of the uncertainty (or randomness) of a pmf. Under the acceptance of Shannon entropy as MoU for pmf, the desideratum D3 makes perfect sense. Of course, this desideratum D3 could be disputed (and eventually rejected) if one can cast in doubt (based on very strong justification) the use of Shannon entropy as MoU for pmf. For alternatives of Shannon entropy, see for instance the non-exhaustive list of alternatives [28-30] and discussions.[9,31-33]

● **Desideratum D4**: (unicity of max value of U(m))

$$\forall m \neq m_v \quad U(m) < U(m_v) \tag{4}$$

where m is any BBA different from the vacuous BBA m_V defined with respect to the same FoD.

Justification of D4: This fourth desideratum is very important and it makes perfect sense also because the total ignorant source of evidence is characterized by the vacuous BBA $m_v(.)$, and no source of evidence can be more uncertain than the total ignorant source, so the unique maximum value of $U(m)$ must be obtained for $U(m_v)$. As it will be shown next, many existing MoUs fail to satisfy this important and essential desideratum.

Effectiveness of a measure of uncertainty: A measure of uncertainty $U(m)$ is said effective if and only if it satisfies desiderata D1, D2, D3, and D4 and if it is strongly well justified. Any MoU that fails to satisfy at least one of these desiderata is said non-effective, and in this case it cannot be considered seriously as a satisfactory measure of uncertainty for characterizing a basic belief assignment of a source of evidence. Consequently, all non-effective MoUs should be discarded in all applications that necessitate some MoU evaluation.

Remark 1: It is worth noting that we do not specify *a priori* what should be the range of an effective MoU in contrary to some axiomatic attempts made by different authors as reported, for instance, in [15,34,35,104]. We consider that the choice of the range must not be chosen a priori. The maximum range must result of the effective MoU mathematical definition. We only request the satisfaction of the desideratum D4, which is much more general, natural, and essential.

Remark 2: We voluntarily do not include the subadditivity desideratum in the list of our desiderata for the search of an effective MoU in the belief function framework because this desideratum appears in general (i.e. for non-Bayesian non-vacuous BBAs) to be incompatible with essential desideratum D4, and thus it is illusory and vain to ask for a sub-additive MoU for non-Bayesian non-vacuous BBAs. We recall that the subadditivity condition is defined by $U(m^{\Theta\times\Theta'})\leq U(m^{\downarrow\Theta})+U(m^{\downarrow\Theta'})$ or any joint BBA defined on the Cartesian product $\Theta\times\Theta'$ of FoDs Θ and Θ', where $m^{\downarrow\Theta}$ is the marginal (i.e. projection) of $m^{\Theta\times\Theta'}(.)$ on the power-set 2^{Θ}, and $m^{\downarrow\Theta'}$ is the marginal (i.e. projection, see[36,37] for definition) of $m^{\Theta\times\Theta'}(.)$ on the power-set $2^{\Theta'}$. This impossibility comes from the fact that there exist in general $2^{|\Theta\times\Theta'|}-2^{|\Theta|}\cdot 2^{|\Theta'|}>0$ elements of the power set $2^{\Theta\times\Theta'}$ (including some disjunctions of elements of $\Theta\times\Theta'$) whose mass of belief cannot be obtained from the masses of elements of 2^{Θ} and of $2^{\Theta'}$, and which contribute in the uncertainty measure of the joint BBA $m^{\Theta\times\Theta'}$. Indeed, if $|\Theta|=N$ and $|\Theta'|=N'$ the Cartesian product space $\Theta\times\Theta'$ has $N\cdot N'$ elements and its power set $2^{\Theta\times\Theta'}$ has $2^{N\cdot N'}$ elements which is always bigger than the Cartesian product space of power sets $2^{\Theta}\times 2^{\Theta'}$ because $2^{N}\cdot 2^{N'}(=2^{N+N'})<2^{N\cdot N'}$ as soon as $N>2$ and $N'>2$. It is worth mentioning also that most of elements of $2^{\Theta}\times 2^{\Theta'}$ do not have the same structure as the elements of the power set $2^{\Theta\times\Theta'}$. This means that we cannot recover the joint BBA $m^{\Theta\times\Theta'}$ from the product, or combination, of its marginal $m^{\downarrow\Theta}$ and $m^{\downarrow\Theta'}$ in general, but if the joint BBA is totally vacuous or if the joint BBA is Bayesian and if it is equal to the product of two so-called non-interacting (or independent) probability measures[8]. To be more clear, consider two FoDs Θ and Θ' with $|\Theta|=2$ and $|\Theta'|=3$. Hence the

Cartesian product space $\Theta\times\Theta'$ has 2 · 3 = 6 elements[1], and its power set $2^{\Theta\times\Theta'}$ has 2^6 = 64 elements (couples, and unions of couples). If we consider the vacuous BBA $m_v^{\Theta\times\Theta'}$ on $2^{\Theta\times\Theta'}$ defined by $m_v^{\Theta\times\Theta'}(\Theta\times\Theta')=1$, then its projection on Θ is the vacuous BBA $m_v^{\Theta}(\Theta)=1$ defined on the FoD $\Theta=\{\theta_1,\theta_2\}$ having only two elements, and its projection on Θ' is the vacuous BBA $m_v^{\Theta'}(\Theta')=1$ defined on the FoD $\Theta'=\{\theta'_1,\theta'_2,\theta'_3\}$ having only three elements. Why the MoU of $m_v^{\Theta\times\Theta'}$ (i.e. full ignorant source) related to 6 elements of $\Theta\times\Theta'$ should be less (or equal) to the sum of MoU of m_v^{Θ} related to only the two elements of Θ and the MoU of $m_v^{\Theta'}$ only related to the three elements of Θ'? To amplify this point, if we consider $|\Theta|=5$ and $|\Theta'|=8$ then $|\Theta\times\Theta'|=40$. Why the MoU of the vacuous BBA $m_v^{\Theta\times\Theta'}$ related to 40 elements of $\Theta\times\Theta'$ should be less (or equal) to the sum of MoU of vacuous BBA m_v^{Θ} related to only 5 elements of Θ and the MoU of the vacuous BBA $m_v^{\Theta'}$ only related to the 8 elements of Θ'? We do not see any solid theoretical reason, nor intuitive reason, for justifying and requiring the subadditivity desideratum in the general framework of belief functions, and put it as a property to be generally satisfied.[15] Unlike Vejnarova and Klir opinions [38(p.28)] and many authors, we do not consider that the meaningful measure of uncertainty of basic belief assignment must satisfy the subadditivity property. The proposal of adding the desiderata of subadditivity, additivity, and monotonicity for a search of a MoU of belief functions had been explored and defended by Klir in[2] at the end of 1980s. It is however worth mentioning that if a MoU satisfies the desideratum D3 (when the BBA is Bayesian), its subadditivity property is always guaranteed because Shannon entropy is subadditive.[8,20]

3. Existing measures of uncertainty

In this section we analyze most of existing measures of uncertainty available in the open literature related to belief functions. We verify if these measure pass, or not, the effectiveness test. We say that a MoU fails the effectiveness test if at least one of the desiderata D1, D2, D3 or D4 is not satisfied by the MoU under test. If necessary, we explain what is the problem with this MoU and when necessary we give a counter-example for it.

Tables 1 and 2 show the formulas of all the MoUs analyzed in this work. Some existing MoUs capture only some aspects of uncertainty [2] and have

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1 Each element is a couple of the form (θ_i,θ'_j), $i=1,2$ and $j=1,2,3$.

2 Referred to as entropy-like uncertainty, non-specificity (or imprecision), and fuzziness which is uniquely connected with fuzzy sets.[10]

specific names given by their authors (e.g. conflict, dissonance, discord, strife, etc.) listed in the third column of these tables.[3] For convenience, the MoUs have been indexed and listed by the year of their publication in Tables 1 and 2. We have also included in Tables 1 & 2 the names of authors of the MoUs, the names of the MoU when it exists (and eventually new names if needed for clarity), and the formulas of the MoUs. For convenience, we have used the natural log in the mathematical expressions of MoUs for the homogeneity of the presentation. Some authors prefer $\log_2$ instead, but this preference does not really matter because the values of an expression will differ only from the constant multiplicative factor 1/log(2), and the unity will just change from nats to bits.

Table 3 indicates if each MoU satisfies, or not, the desiderata D1, D2, D3 and D4, and thus if it passes the effectiveness test, or not. Most of results listed in Table 3 are easy to verify directly from the mathematical definition of each MoU of Tables 1 and 2, and are left as exercises for the reader. Some results however of Table 3, specially those related to the failure of D4 desideratum, may appear less obvious to verify and that is why we give some numerical counter-examples for them in the Tables 4 and 5 for convenience.[4] These counter-examples have been obtained from Monte-Carlo simulation of randomly generated BBAs for testing the desiderata. Of course, many more counter-examples can be found by Monte-Carlo simulation, but of course only one is sufficient to prove the failure of a MoU for a desideratum, specially for D4. Extra justifications about violation of desiderata by some MoUs are presented next.

The $MoU_{1984}(m) = -\sum_{X\subseteq\Theta} m(X)\log(m(X))$ does not satisfy D2 desideratum because $MoU_{1984}(m_v^{\Theta}) = 0$ whatever is the size of the FoD Θ. Consequently, $MoU_{1984}(m) > MoU_{1984}(m_v)$ if $m \neq m_v$, hence D4 desideratum is violated. That is why $MoU_{1984}(m)$ cannot be recommended as an effective measure of uncertainty.

The $MoU_{1990b}(m) = T(m)$ does not satisfy D4 desideratum because we can have $m \neq m_v$ such that $T(m) = T(m_v)$ as shown in the counter-example given in[52(p165)]. See also our simpler counter-example given in Table 4.

The $MoU_{1992}(m) = S(m)$ (i.e. the strife) does not satisfy D2 desideratum because one can easily verify that one has always[5] $S(m_v^{\Theta}) = S(m_v^{\Theta'}) = 0$ when

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[3] The names and notations are not always homogeneous from one author to another, for instance U-uncertainty is also called non-specificity and denoted by N(m).[39-41]

[4] The numerical values have been truncated to their third digit.

[5] It is worth noting that Klir's statement, at the bottom of page 86 of Klir and Wierman,[8] stating (using our notation) that $S(m_V) = \log(|\Theta|)$ is clearly wrong.

$|\Theta| \neq |\Theta'|$. The strife does not satisfy D4 either because if m is the uniform Bayesian BBA on (non-empty) FoD Θ, one has $S(m) = \log(|\Theta|)$ which is greater than zero, proving that S(m) violates D4.

The $MoU_{1992b}(m)$ = NS(m) does not satisfy D4 desideratum because we can have $m \neq m_v$ but such that NS(m) = NS(m_V), as shown in the counter-example of Table 4, where[6] $U(m) = \log(2)$ and $S(m) = \log(3) - \log(2)$, so that $NS(m) = U(m) + S(m) = \log(3)$, and we have U($m_V$) = log(3) and S($m_V$) = 0 yielding NS($m_V$) = log(3), and hence proving NS(m) = NS(m_V).

The $MoU_{1994}(m)$ = AU(m), proposed by Harmanec and Klir,[39,40] is nothing but the maximal Shannon entropy value obtained by analyzing all the pmfs P(·) compatible with Bel(·) and Pl(·) functions of the BBA m(·) such that for all $X \subseteq \Theta$, $Bel(X) \leq \sum_{\theta_i \in X} P(\theta_i) \leq Pl(X)$. More precisely,

$$P^*(\cdot) = \arg \max_{\substack{\text{All compatible} \\ \text{pmf } P(.)}} -\sum_{\theta_i \in \Theta} P(\theta_i) \log(P(\theta_i))$$

This max-entropy pmf $P^*(\cdot)$ is obtained by solving a non-linear optimization problem.[86-88] It is clear that this MoU, as well as all other Shannon-alike entropy measures based on different probabilistic approximations techniques [7] (as BetP-entropy, PlPr-entropy, or DSmP-entropy, etc) of (non-Bayesian) BBA m to a Bayesian BBA fail to satisfy D4 desideratum. Indeed, the vacuous BBA m_V will always be approximated by the uniform pmf $P^{unif}(\cdot)$ defined on the FoD Θ, and there will be no difference between the Shannon-alike entropy value for m_V (for the total ignorant source of evidence) and the Shannon-alike entropy value of the Bayesian uniform BBA. This explains why AU(m) and all other Shannon-alike entropies violate the D4 desideratum.

The $MoU_{1996}(m)$ = TC(m) violates D2 because TC(m_V) = 0 whatever is the dimension of the (non-empty) FoD Θ. It violates D3, because for Bayesian BBA one gets $TC(m) = \sum_{i=1}^{n} P(\theta_i)(1 - P(\theta_i))$ as reported in [45]. It also violates D4 in general because for Bayesian BBA one has $TC(m) > 0$, except in the particular Bayesian case where the BBA is entirely focused on a singleton θ_i (i.e. $m(\theta_i) = 1$). In this particular case we obtain $TC(m) = TC(m_v) = 0$. So for all Bayesian BBAs m we will always have $TC(m) \geq TC(m_v)$, which clearly violates D4 desideratum.

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6 The easy verification from U(m) and S(m) formulas is left to the readers.

7 $BetP_m$, $DSmP_m$ and $PlPr_m$ are different probabilistic transformations of a non-Bayesian BBA into a Bayesian one. They have been proposed by different authors [89-91] providing details.

The original formula of $MoU_{1997}(m) = H_{ds}(m)$ proposed by Maluf,[57] was actually $H_{ds}(m) = -\sum_{X\subseteq\Theta} Pl(X)\log(Bel(X))$ which is obviously ill-defined when $Pl(X) > 0$ and $Bel(X) = 0$ because $\log(0) = -\infty$. That is why we did consider only focal elements of the BBAs m in the modified formula $H_{ds}(m)$ given in Table 1. For any cardinality of non-empty FoD Θ we have always $H_{ds}(m_v) = 0$ because for the vacuous BBA m_v, the only focal element is Θ for which $Bel(\Theta) = Pl(\Theta) = 1$ so that $H_{ds}(m_v) = -Pl(\Theta)\log(Bel(\Theta)) = -1\log(1) = 0$. So, $H_{ds}(m)$ violates D2. This MoU violates also D4 because for Bayesian BBA $H_{ds}(m)$ is the same as Shannon entropy, and Shannon entropy is greater than zero in general.

The $MoU_{2000}(m) = H_S(m)$ (Shapley entropy) coincides with Shannon entropy for Bayesian BBAs, and one can easily verify that $H_S(m_v) = \log(|\Theta|)$ which

Table 1. List of existing MoUs for the period 1980-2000.

Measures of Uncertainty	Author(s) & Ref.	Name	Mathematical expression
$MoU_{1981}(m)$	Höhle[42,43]	confusion	$C(m) = -\sum_{X\subseteq\Theta} m(X)\log(Bel(X))$
$MoU_{1983}(m)$	Yager[44]	dissonance	$E(m) = -\sum_{X\subseteq\Theta} m(X)\log(Pl(X))$
$MoU_{1983b}(m)$	Yager[44,45]	nonspecificity	$N(m) = 1 - \sum_{X\subseteq\Theta} m(X)/\lvert X\rvert$
$MoU_{1983c}(m)$	Dubois[46,47,48]	U-uncertainty	$U(m) = \sum_{X\subseteq\Theta} m(X)\log(\lvert X\rvert)$
$MoU_{1984}(m)$	Höhle[1,49,50]	entropy of discernible-ness	$E_m(m) = -\sum_{X\subseteq\Theta} m(X)\log(m(X))$
$MoU_{1987}(m)$	Dubois et al.[1]	entropy-like index	$C'(m) = -\sum_{X\subseteq\Theta} m(X)\log(q(X))$
$MoU_{1987b}(m)$	Dubois et al.[1]	index of fuzziness	$d(m) = -\log(\sum_{X\subseteq\Theta} m(X)Bel(X))$
$MoU_{1988}(m)$	Dubois et al.[51]	imprecision	$I(m) = \sum_{X\subseteq\Theta} m(X)\lvert X\rvert$
$MoU_{1988b}(m)$	Lamata et al.[37]	*lower* entropy	$L_{ent}(m) = E(m) + U(m) = -\sum_{X\subseteq\Theta} m(X)\log(Pl(X)/\lvert X\rvert)$
$MoU_{1988c}(m)$	Lamata et al.[37]	*upper* entropy	$U_{ent}(m) = -\sum_{X\subseteq\Theta} m(X)\sup\{\log(Pl(\theta_i)) \mid \theta_i \in X\} + \log(\sum_{X\subseteq\Theta} m(X)\lvert X\rvert)$
$MoU_{1990}(m)$	Klir et al.[52,53,54]	discord	$D(m) = -\sum_{X\subseteq\Theta} m(X)\log(\sum_{Y\subseteq\Theta} m(Y)\frac{\lvert X\cap Y\rvert}{\lvert Y\rvert})$
$MoU_{1990b}(m)$	Klir et al.[52,53]	*total* uncertainty	$T(m) = U(m) + D(m)$
$MoU_{1992}(m)$	Klir et al.[38,53]	strife	$S(m) = -\sum_{X\subseteq\Theta} m(X)\log(\sum_{Y\subseteq\Theta} m(Y)\frac{\lvert X\cap Y\rvert}{\lvert X\rvert})$
$MoU_{1992b}(m)$	Klir et al.[53]		$NS(m) = U(m) + S(m)$
$MoU_{1993}(m)$	Pal et al.[6]	*average total* uncer-tainty	$ATU(m) = -\sum_{X\subseteq\Theta} m(X)\log(m(X)) + U(m)$
$MoU_{1993b}(m)$	Maeda et al.[55]	Maeda extended en-tropy	$M(m) = AU(m) + U(m)$
$MoU_{1994}(m)$	Harmanec[39,56]	amount of uncertainty	$AU(m) = -\sum_{\theta_i\in\Theta} P^*(\theta_i)\log(P^*(\theta_i))$
$MoU_{1996}(m)$	George et al.[9,45]	*total* conflict	$TC(m) = \sum_{X\subseteq\Theta} m(X)(\sum_{Y\subseteq\Theta} m(Y)[1 - \frac{\lvert X\cap Y\rvert}{\lvert X\cup Y\rvert}])$
$MoU_{1997}(m)$	Maluf[57]	Maluf entropy	$H_{ds}(m) = -\sum_{X\subseteq\Theta \mid m(X)>0} Pl(X)\log(Bel(X))$
$MoU_{1999}(m)$	Klir[58]	Shannon-like measure	$SL(m) = -\sum_{\theta_i\in\Theta} \frac{Bel(\theta_i)\log(Bel(\theta_i)) + Pl(\theta_i)\log(Pl(\theta_i))}{\sum_{\theta_j\in\Theta} Bel(\theta_j) + Pl(\theta_j)}$
$MoU_{2000}(m)$	Yager[59,60]	Shapley entropy	$H_S(m) = -\sum_{\theta_i\in\Theta}[\sum_{X\subseteq\Theta \mid \theta_i\in X} \frac{m(X)}{\lvert X\rvert}]\log(\sum_{X\subseteq\Theta \mid \theta_i\in X} \frac{m(X)}{\lvert X\rvert})$

Table 2. List of existing MoUs for the period 2001-2021.

Measures of Uncertainty	Author(s) & Ref.	Name	Mathematical expression
$MoU_{2003}(m)$	Dezert[61], Jousselme et al.[11]	Pignistic entropy or BetP-entropy or Ambiguity measure	$AM(m) = -\sum_{\theta_i\in\Theta} BetP_m(\theta_i)\log(BetP_m(\theta_i))$
$MoU_{2016}(m)$	Deng[62]	Deng entropy	$E_d(m) = -\sum_{X\subseteq\Theta} m(X)\log(\frac{m(X)}{2^{\vert X\vert}-1})$
$MoU_{2016b}(m)$	Yang et al.[63]	total uncertainty, $d^I(\cdot,\cdot)$ is Wasserstein distance	$TU^I(m) = 1-\frac{\sqrt{3}}{\vert\Theta\vert}\sum_{\theta_i\in\Theta} d^I([Bel(\theta_i), Pl(\theta_i)], [0,1])$
$MoU_{2017}(m)$	Deng et al.[64]	improved $TU^I(m)$ with Euclidean distance d_E^I	$iTU^I(m) = \sum_{\theta_i\in\Theta} [1-d_E^I([Bel(\theta_i), Pl(\theta_i)], [0,1])]$
$MoU_{2017b}(m)$	Zhou et al.[65,66,67]	improved Deng entropy	$E_{Id}(m) = -\sum_{X\subseteq\Theta} m(X)\log(\frac{m(X)}{2^{\vert X\vert}-1}e^{\frac{\vert X\vert-1}{\vert\Theta\vert}})$
$MoU_{2017c}(m)$	Tang et al.[68]	Tang weighted belief entropy	$E_{Wd}(m) = -\sum_{X\subseteq\Theta}\frac{\vert X\vert}{\vert\Theta\vert} m(X)\log(\frac{m(X)}{2^{\vert X\vert}-1})$
$MoU_{2018}(m)$	Jiroušek et al.[69]	Extended PlPr-entropy	$H^{ext}_{PlPr}(m) = -\sum_{\theta_i\in\Theta} PlPr_m(\theta_i)\log(PlPr_m(\theta_i)) + U(m)$
$MoU_{2018b}(m)$	Jiroušek et al.[70,71]	q-entropy	$H_q(m) = \sum_{X\subseteq\Theta}(-1)^{\vert X\vert} q(X)\log(q(X))$
$MoU_{2018c}(m)$	Mambé et al.[72]	Mambé entropy	$E_{Nm}(m) = -\sum_{X\subseteq\Theta} m(X)\log(\frac{m(X)}{2^{\vert X\vert}-1}e^{\frac{\vert X\vert-1}{2^{\vert\Theta\vert}}})$
$MoU_{2018d}(m)$	Pan et al.[73]	Pan 1st entropy	$H_{bel}(m) = -\sum_{X\subseteq\Theta}\frac{Bel(X)+Pl(X)}{2}\log(\frac{Bel(X)+Pl(X)}{2(2^{\vert X\vert}-1)})$
$MoU_{2018e}(m)$	Wang et al.[74]	Wang entropy	$SU(m) = \sum_{\theta_i\in\Theta}[-\frac{Bel(\theta_i)+Pl(\theta_i)}{2}\log_2(\frac{Bel(\theta_i)+Pl(\theta_i)}{2}) + \frac{Pl(\theta_i)-Bel(\theta_i)}{2}]$
$MoU_{2019}(m)$	Li et al.[75]	Li entropy	$IQ(m) = \sum_{X\subseteq\Theta}(\frac{m(X)}{2^{\vert X\vert}-1})^2$
$MoU_{2019b}(m)$	Cui et al.[76]	Cui entropy	$E_{Cui}(m) = -\sum_{X\subseteq\Theta} m(X)\log(\frac{m(X)}{2^{\vert X\vert}-1}\cdot e^{\sum_{Y\subseteq\Theta, Y\neq X \& m(Y)>0}\frac{\vert X\cap Y\vert}{2^{\vert\Theta\vert}-1}})$
$MoU_{2019c}(m)$	Pan et al.[77]	Pan 2nd entropy	$H_{PQ}(m) = -\sum_{X\subseteq\Theta} m(X)\log(\sum_{\theta_i\in X} PlPr_m(\theta_i)) + U(m)$
$MoU_{2019d}(m)$	Chen et al.[78]	Chen entropy	$E_i(m) = -\sum_{X\subseteq\Theta} m(X)\log(\frac{m(X)}{2^{\vert X\vert}-1}\cdot\frac{\vert X\vert}{\vert\bigcup_{Y\subseteq\Theta, m(Y)>0} Y\vert})$
$MoU_{2019e}(m)$	Zhao et al.[79]	Zhao entropy	$H_{inter}(m) = -\sum_{\theta_i\subseteq\Theta}\frac{Bel(\theta_i)+Pl(\theta_i)}{2}\log(\frac{Bel(\theta_i)+Pl(\theta_i)}{2}e^{-(Pl(\theta_i)-Bel(\theta_i))}) - \sum_{X\subseteq\Theta, \vert X\vert>1} m(X)\log(\frac{m(X)}{2^{\vert X\vert}-1}e^{-(Pl(X)-Bel(X))})$
$MoU_{2020}(m)$	Li et al.[80]	Li improved entropy	$IQ_{Li}(m) = \sum_{X\subseteq\Theta}(\frac{m(X)}{2^{\vert X\vert}-1})^2\cdot e^{\sum_{Y\subseteq\Theta, Y\neq X}\frac{\vert X\cap Y\vert}{\vert\Theta\vert}}$
$MoU_{2020b}(m)$	Wen et al.[81]	Wen entropy	$U_{exp}(m) = \frac{1}{e-\frac{1}{\vert\Theta\vert^2}e^{\frac{1}{\vert\Theta\vert^2}}}[e-\sum_{X\subseteq\Theta, \vert X\vert=1} m(X)e^{m(X)} - \sum_{X\subseteq\Theta, \vert X\vert\neq 1}\frac{m(X)}{\vert X\vert\cdot\vert\bigcup_{Y\subseteq\Theta, m(Y)>0} Y\vert}e^{\frac{m(X)}{\vert X\vert\cdot\vert\bigcup_{Y\subseteq\Theta, m(Y)>0} Y\vert}}]$
$MoU_{2020c}(m)$	Chen et al.[82]	Chen improved entropy	$E^C_{Wd}(m) = -\sum_{X\subseteq\Theta}\frac{\vert X\vert}{\vert\Theta\vert}\frac{1-m(X)}{\vert\mathcal{F}_\Theta(m)\vert-1}\log(\frac{1-m(X)}{\vert\mathcal{F}_\Theta(m)\vert-1}\cdot\frac{1}{2^{\vert X\vert}-1})$
$MoU_{2020d}(m)$	Qin et al.[83]	Qin entropy	$Q(m) = E_m(m) + \sum_{X\subseteq\Theta}\frac{\vert X\vert}{\vert\Theta\vert} m(X)\log(\vert X\vert)$
$MoU_{2020e}(m)$	Yan et al.[84]	Yan entropy	$H_n(m) = -\sum_{X\subseteq\Theta} m(X)\log(\frac{m(X)+Bel(X)}{2}\cdot\frac{1}{2^{\vert X\vert}-1}\cdot e^{\frac{\vert X\vert-1}{\vert\mathcal{C}(m)\vert}})$
$MoU_{2020f}(m)$	Li et al.[85]	Li-Pan entropy	$H_{BF}(m) = E_m(m) + \vert\Theta\vert\cdot U(m)$
$MoU_{2021}(m)$	This paper	Extended BetP-entropy	$H^{ext}_{BetP}(m) = -\sum_{\theta_i\in\Theta} BetP_m(\theta_i)\log(BetP_m(\theta_i)) + U(m)$
$MoU_{2021b}(m)$	This paper	Extended DSmP-entropy	$H^{ext}_{DSmP}(m) = -\sum_{\theta_i\in\Theta} DSmP_m(\theta_i)\log(DSmP_m(\theta_i)) + U(m)$

is also the same maximum value of Shannon entropy for the uniform Bayesian BBA. Hence, $H_S(m)$ is not the unique maximum measure of uncertainty value when we use Shapley entropy. It can also be verified that this maximum value can be also obtained by non-Bayesian BBA. For instance, if $\Theta=\{\theta_1,\theta_2,\theta_3\}$ and $m(\theta_1\cup\theta_2)=m(\theta_1\cup\theta_3)=m(\theta_2\cup\theta_3)=1/3$, then $H_S(m)=\log(3)$, which is also the same value as for $H_S(m_v^\Theta)$. Because Shapley entropy proposed by Yager violates D4 desideratum, we cannot recommend it as an effective MoU.

Table 3. Desiderata verification, and effectiveness test results.

Measures of Uncertainty	D1 min U(m) = 0 for $m(\theta_i) = 1$	D2 $U(m_v^{\Theta}) < U(m_v^{\Theta'})$ if $\lvert\Theta\rvert < \lvert\Theta'\rvert$	D3 U(m) is Shannon entropy for Bayesian BBA	D4 $U(m) < U(m_v)$ if $m \neq m_v$	Effectiveness test result
$MoU_{1981}(m) = C(m)$	yes	**no**	yes	**no**	failed
$MoU_{1983}(m) = E(m)$	yes	**no**	yes	**no**	failed
$MoU_{1983b}(m) = N(m)$	yes	yes	**no**	yes	failed
$MoU_{1983c}(m) = U(m)$	yes	yes	**no**	yes	failed
$MoU_{1984}(m) = E_m(m)$	yes	**no**	yes	**no**	failed
$MoU_{1987}(m) = C'(m)$	yes	**no**	yes	**no**	failed
$MoU_{1987b}(m) = d(m)$	yes	**no**	**no**	**no**	failed
$MoU_{1988}(m) = I(m)$	**no**	yes	**no**	yes	failed
$MoU_{1988b}(m) = L_{ent}(m)$	yes	yes	yes	**no**	failed
$MoU_{1988c}(m) = U_{ent}(m)$	yes	yes	yes	**no**	failed
$MoU_{1990}(m) = D(m)$	yes	**no**	yes	**no**	failed
$MoU_{1990b}(m) = T(m)$	yes	yes	yes	**no**	failed
$MoU_{1992}(m) = S(m)$	yes	**no**	yes	**no**	failed
$MoU_{1992b}(m) = NS(m)$	yes	yes	yes	**no**	failed
$MoU_{1993}(m) = ATU(m)$	yes	yes	yes	**no**	failed
$MoU_{1993b}(m) = M(m)$	yes	yes	yes	yes	**okay**
$MoU_{1994}(m) = AU(m)$	yes	yes	yes	**no**	failed
$MoU_{1996}(m) = TC(m)$	yes	**no**	**no**	**no**	failed
$MoU_{1997}(m) = H_{ds}(m)$	yes	**no**	yes	**no**	failed
$MoU_{1999}(m) = SL(m)$	yes	**no**	yes	**no**	failed
$MoU_{2000}(m) = H_S(m)$	yes	yes	yes	**no**	failed
$MoU_{2003}(m) = AM(m)$	yes	yes	yes	**no**	failed
$MoU_{2016}(m) = E_d(m)$	yes	yes	yes	**no**	failed
$MoU_{2016b}(m) = TU^I(m)$	yes	**no**	**no**	yes	failed
$MoU_{2017}(m) = iTU^I(m)$	yes	yes	**no**	yes	failed
$MoU_{2017b}(m) = E_{Id}(m)$	yes	yes	yes	**no**	failed
$MoU_{2017c}(m) = E_{Wd}(m)$	yes	yes	**no**	**no**	failed
$MoU_{2018}(m) = H_{PIPr}^{ext}(m)$	yes	yes	yes	yes	**okay**
$MoU_{2018b}(m) = H_q(m)$	**no**	**no**	yes	**no**	failed
$MoU_{2018c}(m) = E_{Nm}(m)$	yes	yes	yes	**no**	failed
$MoU_{2018d}(m) = H_{bel}(m)$	**no**	yes	**no**	**no**	failed
$MoU_{2018e}(m) = SU(m)$	yes	yes	yes	yes	**okay**
$MoU_{2019}(m) = IQ(m)$	yes	**no**	**no**	**no**	failed
$MoU_{2019b}(m) = E_{Cui}(m)$	yes	yes	yes	**no**	failed
$MoU_{2019c}(m) = H_{PQ}(m)$	yes	yes	yes	**no**	failed
$MoU_{2019d}(m) = E_i(m)$	yes	yes	**no**	**no**	failed
$MoU_{2019e}(m) = H_{inter}(m)$	yes	yes	yes	**no**	failed
$MoU_{2020}(m) = IQ_{Li}(m)$	**no**	yes	**no**	**no**	failed
$MoU_{2020b}(m) = U_{exp}(m)$	yes	**no**	**no**	yes	failed
$MoU_{2020c}(m) = E_{Wd}^{C}(m)$	**no** (NaN)	**no** (NaN)	**no**	**no** (NaN)	failed
$MoU_{2020d}(m) = Q(m)$	yes	yes	yes	**no**	failed
$MoU_{2020e}(m) = H_n(m)$	yes	yes	**no**	**no**	failed
$MoU_{2020f}(m) = H_{BF}(m)$	yes	yes	yes	**no**	failed
$MoU_{2021}(m) = H_{BetP}^{ext}(m)$	yes	yes	yes	yes	**okay**
$MoU_{2021b}(m) = H_{DSmP}^{ext}(m)$	yes	yes	yes	yes	**okay**

The $MoU_{2016}(m) = E_d(m)$ (Deng entropy) has recently aroused the interest and enthusiasm of some researchers because it was highly publicized by Deng during the last five years.[14] We really wonder about such strong interest of this MoU because Deng entropy is obviously not effective, as proved by our simple counter-example given in Table 5. Abellán has already pointed out the problem of Deng entropy.[92] Nevertheless, some researchers try to use it, publicize it or improve it unsuccessfully as shown in our analysis summarized in Table 3. So, it is clear that Deng Entropy is not recommended for applications, as well as other generalizations (modifications or extensions) of it, as those recently proposed by the same author (Rényi-Deng (R-D) entropy, Tsallis-Deng (T-D) entropy, Rényi-Tsallis-Deng (R-T-D) entropy, Interval-valued Deng entropy, Fractal-based belief Deng entropy, Deng entropy for orderable set, etc.; see for instance [93,94]) because they do not have real interest since they are non-effective. We emphasize that even if a MoU collapses with Shannon entropy (as Deng entropy does) when a BBA is Bayesian, it can be non-effective and useless if it violates D4 desideratum. That is why Deng entropy (and all its recent variants based on it) is not effective as most of other MoUs actually reported in Table 3.

The $MoU_{2018b}(m) = H_q(m)$ (q-entropy alike) violates D1 because $H_q(m)$ can be negative so its minimum value is not zero. For instance if $\Theta=\{\theta_1,\theta_2,\theta_3\}$ and $m(\theta_1\cup\theta_2)=m(\theta_1\cup\theta_3)=m(\theta_2\cup\theta_3)=1/3$, then $H_q(m)\approx-0.2877$. This MoU also violates D2 because $H_q(m_v)=0$ whatever is the dimension of the (non-empty) FoD Θ. This MoU collapses with Shannon entropy because if m is a Bayesian BBA one has $q(X)=m(X)$ for all $X\subseteq\Theta$, and the focal elements of m are necessarily singletons $X\subseteq\Theta$ for which $|X|=1$, so that $(-1)^{|X|}=-1$, and consequently the mathematical definition of $H_q(m)$ given in Table 1 is same as Shannon entropy. This MoU violates D4 because for Bayesian BBA $H_q(m)$ is the same as Shannon entropy, and Shannon entropy is greater than zero in general[8]. For instance if $\Theta=\{\theta_1,\theta_2,\theta_3\}$ and $m(\theta_1)=m(\theta_2)=m(\theta_3)=1/3$, then $H_q(m)=\log(|\Theta|)=\log(3)>0$. Hence $H_q(m)>H_q(m_v)$.

The $MoU_{2018d}(m) = H_{bel}(m)$ (Pan 1st entropy) violates D1 because if we consider the simplest case of FoD with $\Theta=\{\theta_1,\theta_2\}$, and the specific BBA $m(\theta_1)=1$, we have $[Bel(\theta_1),Pl(\theta_1)]=[1,1]$, $[Bel(\theta_2),Pl(\theta_2)]=[0,0]$ and $[Bel(\theta_1\cup\theta_2),Pl(\theta_1\cup\theta_2)]=[1,1]$ so we have $(Bel(\theta_1)+Pl(\theta_1))/2=1$, $(Bel(\theta_2)+Pl(\theta_2))/2=0$ and $(Bel(\theta_1\cup\theta_2)+Pl(\theta_1\cup\theta_2))/2=1$. Hence $H_{bel}(m)=-1\log(1/(2^1-1))-0\log(1/(2^1-1))-1\log(1/(2^2-1))=\log(3)>0$. Pan 1st entropy violates D3 (Shannon entropy consistency) too, because if m is the uniform Bayesian BBA given by $m(\theta_1)=m(\theta_2)=0.5$, then $H_{bel}(m)=-0.5\log(0.5)-0.5\log(0.5)-1\log(1/3)=\log(2)+\log(3)$ which is greater than Shannon entropy which is equal to $-0.5\log(0.5)-0.5\log(0.5)=\log(2)$. Pan 1st entropy violates D4 also because for the vacuous BBA $m_v(\theta_1\cup\theta_2)=1$, one

8 Except in the case where $m(\theta_i) = 1$ for some $\theta_i \in \Theta$.

has
$[Bel(\theta_1),Pl(\theta_1)]=[0,1],[Bel(\theta_2),Pl(\theta_2)]=[0,1]$ and $[Bel(\theta_1\cup\theta_2),Pl(\theta_1\cup\theta_2)]=[1,1]$

and $(Bel(\theta_1)+Pl(\theta_1))/2=0.5, (Bel(\theta_2)+Pl(\theta_2))/2=0.5$
and $(Bel(\theta_1\cup\theta_2)+Pl(\theta_1\cup\theta_2))/2=1$,
so that $H_{bel}(m)=-0.5\log(0.5)-0.5\log(0.5)-1\log(1/3)=\log(2)+\log(3)$, which is the same value as for uniform Bayesian BBA, so $H_{bel}(m_v)$ is not strictly greater than other Pan 1st entropy values.

The formula of $MoU_{2018e}(m)$ = $SU(m)$ (Wang entropy) has been kept with its original formulation (with $\log_2(\cdot)$ function) in Table 2, so it is expressed in bits. If one wants to express $SU(m)$ in nats we must replace $\log_2(\cdot)$ function by the natural logarithm function $\log(\cdot)$ and the second terms $(Pl(\theta_i)-Bel(\theta_i))/2$ must be multiplied by $\log(2)$ in the mathematical definition of $SU(m)$.

For the $MoU_{2019b}(m)$ = $E_{Cui}(m)$ (Cui entropy) proposed in[76], it is clear that the original mathematical definition of this entropy does not fit with the derivations of what the authors have in mind when making their numerical examples in their paper because of a mistake in their exponential term. That is why we have to correct this term by replacing $\sum_{\substack{Y\subseteq\Theta\\ Y\neq X}}$ by $\sum_{\substack{Y\subseteq\Theta\\ Y\neq X \& m(Y)>0}}$ in the original formula. Cui entropy violates D4 desideratum as shown in the example of Table 5.

The $MoU_{2019c}(m)$ = $H_{PQ}(m)$ (Pan 2nd entropy) is not effective because $H_{PQ}(m_v)$ coincides with $H_{PQ}(m)$ when m is the uniform Bayesian BBA, so it violates D4 desideratum.

The $MoU_{2019d}(m)$ = $E_i(m)$ (Chen entropy) is not effective because one can have $E_i(m)>E_i(m_v)$. For instance, consider the vacuous BBA m_v on FoD $\Theta=\{\theta_1,\theta_2,\theta_3\}$, then $E_i(m_v)=\log(2^{|\Theta|}-1)=\log(7)=1.9459$, and if one considers the uniform Bayesian BBA for which $m(\theta_1)=m(\theta_2)=m(\theta_3)=1/3$ one gets $E_i(m)=-\log(\frac{1}{3}.\frac{1}{3})=2\log(3)=2.1972>E_i(m_v)$. So, Chen entropy violates D4 desideratum.

The $MoU_{2019e}(m)$ = $H_{inter}(m)$ (Zhao entropy) is not effective because it violates D4 desideratum. As a simple counter-example, consider $\Theta=\{\theta_1,\theta_2,\theta_3\}$ with the BBA $m(\theta_1\cup\theta_2)=m(\theta_1\cup\theta_3)=m(\theta_2\cup\theta_3)=1/3$, then $H_{inter}(m)=4.6291$ nats, where for vacuous BBA $m_v(\Theta)=1$ we get $H_{inter}(m_v)=4.4856$ nats. Clearly,

$H_{inter}(m) > H_{inter}(m_v)$ which does not make sense because the vacuous BBA m_v characterizes the most ignorant source of evidence.

It is worth mentioning that the numerical examples given by Li and Cui in their paper are incorrect because they are inconsistent with their original new entropy formula (12) for IQ_{mi}.[80] If we admit that the original Li's definition of entropy is correct then we get the effectiveness test results listed for this entropy in Table 3, and we conclude that the $MoU_{2020}(m) = IQ_{Li}(m)$ (Li improved entropy) is not effective. If we consider that numerical examples by Li and Cui are correct, then we need to modify the exponent term in the original Li's definition (12) of IQ_{mi} as $\sum_{\substack{Y\subseteq\Theta \\ Y\neq X \& m(Y)>0}} |X\cap Y|/|\Theta|$. In this case the effectiveness test result is worse because this modified Li improved entropy will fail to pass the four desiderata, and it is still non-effective.

The $MoU_{2020b}(m) = U_{exp}(m)$ (Wen entropy) violates clearly Shannon entropy compatibility desideratum D2, and for the vacuous BBA m_v one has always $U_{exp}(m_v) = 1$ whatever is the dimension of the FoD Θ. Therefore Wen entropy does not verify desideratum D2. It is not certain that $U_{exp}(m)$ satisfies, or not, D4 desideratum, but we did thousands of Monte Carlo tests with random BBAs for different size of FoD Θ, and $U_{exp}(m)$ did always pass successfully the D4 test, so we conjecture that Wen entropy satisfies D4. Even if our conjecture about satisfaction of D4 for $U_{exp}(m)$ is wrong, it does not change our conclusion that Wen entropy is not effective because it fails to verify D2 and D3.

$MoU_{2020c}(m) = E^C_{Wd}(m)$ (Chen improved entropy) is not mathematically well-defined because when the BBA m has only one focal element (i.e. $|F_\Theta(m)| = 1$), then one has a division by $|F_\Theta(m)| - 1 = 0$ which yields a NaN (Not a Number) indeterminate value in Table 3. Even if $|F_\Theta(m)| > 1$ this entropy is not compatible with Shannon entropy for Bayesian BBAs. So, Chen improved entropy is not effective.

$MoU_{2020d}(m) = Q(m)$ (Qin entropy) violates D4 desideratum because Qin entropy takes same value $\log(|\Theta|)$ for the vacuous BBA and for the uniform Bayesian BBA.

$MoU_{2020e}(m) = H_n(m)$ (Yan entropy) is non-effective. A counter-example for D4 desideratum is given in Table 5 expressed in nats. To express them in bits we have of course to divide our results by log(2). It is worth noting that in Section III.B of[84], the numerical results given by Yan and Deng for $H_n(m_3)$ and $H_n(m_4)$ for their example 5 are wrong.

$MoU_{2020f}(m)=H_{BF}(m)$ (Li-Pan entropy) is also non-effective. A counter-example for D4 desideratum is given in Table 4.

Table 4. Counter-examples for some MoUs.

Elem. of 2^Θ	m	m	m	m	m	m
$\varnothing$	0	0	0	0	0	0
θ_1	1/3	0.18	0	0	0	0.22
θ_2	1/3	0.17	0	0	0	0.09
$\theta_1 \cup \theta_2$	0	0.32	1/3	1/3	1/3	0.02
θ_3	1/3	0.31	0	0	0	0.01
$\theta_1 \cup \theta_3$	0	0	1/3	1/3	1/3	0.11
$\theta_2 \cup \theta_3$	0	0	1/3	1/3	1/3	0.17
$\theta_1 \cup \theta_2 \cup \theta_3$	0	0.02	0	0	0	0.38
MoU(m)	$L_{ent}(m) = 1.098$	$U_{ent}(m) = 1.105$	T(m) = log(3)	NS(m) = log(3)	ATU(m) = 1.791	$H_{BF}(m) = 3.462$
MoU(m_v)	$L_{ent}(m_v) = 1.098$	$U_{ent}(m_v) = 1.098$	T(m_v) = log(3)	NS(m_v) = log(3)	ATU(m_v) = 1.098	$H_{BF}(m_v) = 3.295$

Table 5. Counter-examples for some MoUs (continued).

Elem. of 2^Θ	m	m	m	m	m	m
$\varnothing$	0	0	0	0	0	0
θ_1	0	0	0.10	0	0.11	0.12
θ_2	0	0	0.10	0	0.26	0.09
$\theta_1 \cup \theta_2$	1/3	1/3	0.16	1/3	0.24	0.17
θ_3	0	0	0.03	0	0.01	0.04
$\theta_1 \cup \theta_3$	1/3	1/3	0.06	1/3	0.04	0.15
$\theta_2 \cup \theta_3$	1/3	1/3	0.21	1/3	0.15	0.23
$\theta_1 \cup \theta_2 \cup \theta_3$	0	0	0.34	0	0.19	0.20
MoU(m)	$E_d(m) = 2.197$	$E_{Id}(m) = 1.863$	$E_{Wd}(m) = 2.058$	$E_{Nm}(m) = 2.072$	$H_n(m) = 1.795$	$E_{Cui}(m) = 2.003$
MoU(m_v)	$E_d(m_v) = 1.945$	$E_{Id}(m_v) = 1.279$	$E_{Wd}(m_v) = 1.945$	$E_{Nm}(m_v) = 1.695$	$H_n(m_v) = 1.279$	$E_{Cui}(m_v) = 1.945$

4. Discussion

Our analysis of forty-five measures of uncertainty covering 40 years of research in this field reveals that almost 89 % of them are non-effective because they violate at least one of the four very essential desiderata D1, D2, D3 or D4. In our analysis only five MoUs (M(m) 1993, $H^{ext}_{PIPr}(m)$, 2018, SU(m) 2018, $H^{ext}_{BetP}(m)$, 2021, $H^{ext}_{DSmP}(m)$, 2021) pass successfully the effectiveness test as we can observe in Table 3. We see that all these effective MoUs share two basic principles: 1) approximate the BBA m by a probability measure (i.e. a Bayesian BBA) P_m based on some method and evaluate its Shannon entropy to estimate the randomness (or conflict) inherent to the BBA, and 2) add a term to Shannon entropy value that estimates the level of ambiguity (or non-specificity) inherent of the BBA (usually thanks to Dubois & Prade U-uncertainty). This general principle is simple and quite intuitive but it lacks seriously of theoretical justification. We consider such type of effective MoU construction is unfortunately conceptually flawed and not very satisfactory for the two following reasons.

● **1st reason**: These effective MoUs highly depend on the choice of the method of approximation. This mechanism appears quite arbitrary, and we do no see any strong justification for preferring one of them, either P^* in M(m) MoU, BetP in $H^{ext}_{DSmP}(m)$ MoU, DSmP in $H^{ext}_{DSmP}(m)$ MoU, mid belief-interval value $(Bel(\theta_i)+Pl(\theta_i))/2$ in SU(m) MoU, etc. Worse, a method of approximation can be totally misleading as for instance Cobb-Shenoy $PlPr_m$ transformation[90] because the evaluation of probabilities can be inconsistent with belief interval values. More precisely, one can have $PlPr_m(\theta_i)\notin[Bel(\theta_i),Pl(\theta_i)]$ with Cobb-Shenoy method, which is obviously not reasonable, nor acceptable at all. As a simple counter-example of Cobb-Shenoy transformation, just consider $\Theta=\{\theta_1,\theta_2,\theta_3\}$ with $m(\theta_1)=0.2$ and $m(\theta_2\cup\theta_3)=0.8$. Then, $[Bel(\theta_1),Pl(\theta_1)]=[0.2,0.2]$, $[Bel(\theta_2),Pl(\theta_2)]=[0,0.8]$ and $[Bel(\theta_3),Pl(\theta_3)]=[0,0.8]$ Applying $PlPr_m$ transformation, we get $PlPr_m(\theta_1)=0.2/(0.2+0.8+0.8)\approx 0.112$. Therefore $PlPr_m(\theta_1)<Bel(\theta_1)$ which shows that $PlPr_m(\theta_1)\notin[Bel(\theta_1),Pl(\theta_1)]$. We emphasize the fact that if a method of approximation of a BBA m by a probability measure P_m is chosen, it must be at least consistent with belief interval values generated by the BBA m under concern. Clearly, we cannot recommend Cobb-Shenoy $PlPr_m$ transformation for building an effective MoU based on aforementioned principles 1) and 2) as $H^{ext}_{PlPr}(m)$ MoU proposed recently by Jiroušek and Shenoy based on questionable Shafer semantics and fallacious Dempster's rule arguments.

● **2nd reason**: More fundamentally, we do not see any serious reason which necessitates the arbitrary use of an approximation of any (non-Bayesian) BBA by a Bayesian BBA at first for using Shannon entropy measure as 1st valid principle. Also why do we need, or request, to make the distinction of the two aspects of uncertainty (conflict and non-specificity) in additive manner? This is conceptually very disputable because the randomness (or conflict) and ambiguity (or non-specificity) are actually interwoven in a subtle way that needs to be explored in deep for a better understanding of the mechanism governing the uncertainty with a better description of the (probably non-additive) link between them.

Very recently however Zhang et al. [104] did propose three new innovant effective MoUs not based on arbitrary approximation of the BBA by a probability as in the aforementioned effective MoUs. These measures are denoted by $H^1(m)$, $H^2(m)$ and $H^3(m)$ and respectively defined by [9]

$$H^1(m) = -\sum_{X\subseteq\Theta} m(X)\log_2(Pl(X)) + \sum_{X\subseteq\Theta} m(X) 2\log_2(|X|) \tag{5}$$

$$H^2(m) = -\sum_{X\subseteq\Theta} m(X)\log_2(Pl(X)) + \sum_{X\subseteq\Theta} m(X)\log_2(2^{|X|}-1) \tag{6}$$

$$H^3(m) = -\sum_{X\subseteq\Theta} m(X)\log_2(Pl(X)) + \sum_{\substack{X\subseteq\Theta\\|X|>1}} m(X)|X| \tag{7}$$

These new effective MoUs differ conceptually from the previous effective MoUs $M(m)$, $H^{ext}_{PlPr}(m)$, $SU(m)$, $H^{ext}_{BetP}(m)$ and $H^{ext}_{DSmP}(m)$ but the authors fail to capture well the interwoven link between conflict and non-specificity (or imprecision). Actually the authors set arbitrarily the range of their MoU as a simple parameter, either taken as $\left[0, 2\log_2(|\Theta|)\right]$, $\left[0, \log_2(2^{|\Theta|}-1)\right]$ or $\left[0, |\Theta|\right]$, to define their $H^1(m)$, $H^2(m)$ and $H^3(m)$ measures of uncertainty. This approach is rather ad-hoc and very questionable and possibly other ranges could have been chosen instead. The authors do not identify (or propose) the best MoU to select between $H^1(m)$, $H^2(m)$ and $H^3(m)$ which is a serious problem for using them in applications. Which one to choose? The other serious problem in this approach is the lack of solid justification for using the plausibility function in the summation $-\sum_{X\subseteq\Theta} m(X)\log_2(Pl(X))$. Although effective, these three new MoUs are actually ill-justified and heuristically defined, and somehow they can be considered as conceptually flawed.

5. Conclusion

In this paper we have clearly proved that most of existing measures of uncertainty proposed during the last forty years are actually non-effective, and we consider that the effective ones are conceptually defective. We emphasize the fact that in this jungle of non-effective measures, many of them have bloomed like mushrooms since 2016 with the publication of Deng's paper because of its high publicity. Most of papers since 2016 do not pay attention to the four essential properties that an effective MoU must satisfy, which is a serious problem. We regret this matter of fact, and we hope that this paper has pointed

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9 We correct here the definition of $H^3(m)$ which is mathematically badly formulated by Zhang et al.[104]

out clearly this concern, and also that it will help to reduce the proliferation of useless publications about non-effective MoUs. We encourage the future authors working on new MoUs to verify the effectiveness of their MoU as done recently by Zhang et al.[104] We agree with Abellán, Mantas and E. Bossé vision that an (effective) MoU should not be too complicate to calculate (with direct simple explicit mathematical formula), must obviously incorporate the two aspects of uncertainty (in a subtle and efficient interwoven manner), and must be sensitive to changes of evidence. At last any effective MoU must be conceptually strongly well-justified. This is our roadmap for a search of better effective measures of uncertainty. Is there a better conceptual effective measure of uncertainty for the basic belief assignments? This is a very challenging question. We think that the development of new effective MoU not based on the additive decomposition of conflict and non-specificity is possible as attempted recently by Zhang et al., and we hope that it will appear in a close future.

Appendix 1: Shannon entropy

Consider a random variable represented by a probability mass function (pmf) $P_N = (p_1, p_2, ..., p_N)$, where $p_i = P(\theta_i)$ is the probability of the i-th state θ_i (i.e. outcome) of $\Theta = \{\theta_1, \theta_2, ..., \theta_N\}$. Shannon was interested in communication systems where the various events were the carriers of coded messages, and he did propose (and justify) his entropy measure as appropriate measure of average uncertainty (or measure of randomness) of a random variable[17,18,21,22]. In the classical information theory, the entropy of a random variable is the average level of *surprisal*, or *uncertainty* inherent in the variable's possible outcomes.[95] It is worth noting that Shannon theory does not concern the semantic aspects of the content of a message,[46,96,97] but only its transmission through communication systems. Shannon entropy formula is defined by

$$H(P_N) = -\sum_{i=1}^{|\Theta|} P(\theta_i)\log(P(\theta_i)) \tag{8}$$

By convention, we take $P(\theta_i)\log(P(\theta_i)) = 0$ if $P(\theta_i) = 0$ which is easily justified by continuity since $x\log(x) \to 0$ as $x \to 0$. Adding terms of zero probability does not change the entropy. In (8) we use the natural logarithm (i.e. base *e* logarithm) and in this case, the Shannon entropy value is expressed in *nats* unity. We can also use the base 2 logarithm ($\log_2$) function instead of the natural logarithm, and if so the Shannon entropy value will be expressed in *bits*. In this case, the entropy is the number of bits on average required to describe the random variable, or equivalently the minimum expected number of binary questions required to determine the value of the random variable.

Shannon entropy can be interpreted as a generalization of Hartley entropy (1928) [98,99] when presuming the pmf of equally probable states (i.e. uniform pmf P_N^{unif} for which $P(\theta_i)=1/N$ for $i=1,2,...,N$), hence getting $H(P_N^{unif})=\log(|\Theta|)=\log(N)$. Note that if we have a uniform pmf P_N^{unif} defined on Θ with $|\Theta|=N$ and another uniform pmf $P_{N'}^{unif}$ defined on Θ' with $|\Theta'|=N'$, and if $|\Theta|<|\Theta'|$ then $H(P_N^{unif})<H(P_{N'}^{unif})$ because $\log(|\Theta|)<\log(|\Theta'|)$ since $\log(x)$ is an increasing function. The minimum value of Shannon entropy is zero, which characterizes a *non-random* (or sure) event θ_j for which $P(\theta_j)=1$, because – $-\sum_{i=1}^{|\Theta|}P(\theta_i)\log(P(\theta_i))=-P(\theta_j)\log(P(\theta_j))=0$.

In fact, Shannon rarely used the term information (nor information content) in his works, and he preferred the term entropy to describe the scattering of symbols in the communication system. As reported by Tribus and McIrvine,[100] in 1961 Shannon explained to Tribus his choice for naming the measure of uncertainty as *entropy*, instead of *information* as follows: "My greatest concern was what to call it. I thought of calling it 'information,' but the word was overly used, so I decided to call it 'uncertainty.' When I discussed it with John von Neumann, he had a better idea. Von Neumann told me, 'You should call it entropy, for two reasons. In the first place your uncertainty function has been used in statistical mechanics under that name, so it already has a name. In the second place, and more important, no one really knows what entropy really is, so in a debate you will always have the advantage." Shannon did not prove that his entropy formula is the best measure of uncertainty, and even if it is a measure for information. He only stated a set of reasonable criteria [101] to describe a measure that would serve the requirements of his signal transmission theory, and he found that the entropy formula meets those criteria. We prefer to interpret Shannon entropy as a measure of uncertainty (or randomness) of a pmf, rather than a measure of information content,[101] because of multiple possible interpretations and definitions of information.

The main algebraic properties of the Shannon entropy are (see [20, p.30] for details): the symmetry, the normality,[10] expansibility, decisivity, additivity and recursivity. We recall that Shannon entropy value $H(P_N)$ is always smaller than $H(P_N^{unif})$ if $P_N\neq P_N^{unif}$, expressing the fact that the uniform pmf is the only pmf giving the maximal Shannon entropy value, and characterizing the maximum of

[10] This stipulates that $H(P_2^{unif})=1$ using base 2 logarithm function in (8).

uncertainty (or randomness), which is called the maximality property. Another important property of Shannon entropy is its subadditivity property when considering two (not necessarily independent) events,[20(p.36)] which can be formulated by the following inequality

$$H(P_{N.N'}) \leq H(P_N) + H(P_{N'}) \tag{9}$$

Where $P_{N.N'}$ is the joint pmf defined on Cartesian product space $\Theta \times \Theta' = \{(\theta_i, \theta'_j), i = 1,2,...,N, j = 1,2,...,N'\}$. P_N and $P_{N'}$ are marginal pmfs (i.e. the projections) of the joint pmf $P_{N.N'}$ on spaces (i.e. frames of discernments) Θ and Θ' respectively.

Appendix 2: Belief functions

The belief functions (BF) have been introduced by Shafer [23] to model epistemic uncertainty to reason about uncertainty. We assume that the answer of the problem under concern belongs to a known finite discrete frame of discernement (FoD) $\Theta = \{\theta_1, \theta_2, ..., \theta_N\}$, with $n > 1$, and where all elements of Θ are exhaustive and exclusive. The set of all subsets of Θ (including empty set $\emptyset$, and Θ) is the power-set of Θ denoted by 2^Θ. The number of elements (i.e. the cardinality) of 2^Θ is $2^{|\Theta|}$. A basic belief assignment (BBA) associated with a given source of evidence is a mapping $m(.): 2^\Theta \to [0,1]$ satisfying $m(\emptyset) = 0$ and $\sum_{A \in 2^\Theta} m(A) = 1$. The number $m(A)$ is called the mass of A committed by the source of evidence. The subset $A \in 2^\Theta$ is called a focal element (FE) of the BBA $m(.)$ if and only if $m(A) > 0$. The set of all the focal elements of the BBA $m(.)$ is noted by $F_\Theta(m) = \{X \in 2^\Theta \mid m(X) > 0\}$, or just F for shortand notation when there is no ambiguity on the FoD Θ and the BBA m we are using. The core $C(m)$ of a BBA m is the union of all its focal elements, i.e $C(m) = \bigcup_{X \in F_\Theta(m)} X$.

The belief of A denoted Bel(A) and the plausibility of A denoted Pl(A) are usually interpreted respectively as lower and upper bounds of an unknown (subjective) probability measure P(A). They are respectively defined for any $A \in 2^\Theta$ from the BBA $m(.)$ by

$$Bel(A) = \sum_{X \in 2^\Theta | X \subseteq A} m(X) \tag{10}$$

$$Pl(A) = \sum_{X \in 2^\Theta | A \cap X \neq \emptyset} m(X) = 1 - Bel(\bar{A}) \tag{11}$$

where $\bar{A}$ represents the complement of A in Θ, that is $\bar{A}=\Theta\setminus\{A\}=\{X \mid X\in\Theta \text{ and } X\notin A\}$. The symbol \ denotes the set difference operator. Also, the commonality function $q(\cdot)$ defined for all $A\subseteq\Theta$ by $q(A)=\sum_{X\subseteq\Theta|A\subseteq X} m(X)$ is involved in the some derivations, for instance in the definition of $MoU_{1987}(m)$ (cf Table 1). The vacuous BBA (VBBA for short) representing a totally ignorant source is defined by $m_v(\Theta)=1$. In this short presentation, we implicitly work on the FoD Θ and so we did omit to refer to it in our previous notations. If we have to work with BBAs defined on different FoDs, say Θ and Θ', then we will explicitly indicate these FoDs in the BBA notations as $m^{\Theta}(.)$ and $m^{\Theta'}(.)$. In the classical theory of belief functions the combination of several distinct sources of evidence characterized by their BBAs defined on the same FoD is done with Dempster's rule of combination, see the work of Shafer.[23] To circumvent the problems of Dempster's rule (e.g. its dictatorial behavior, its possible insensitivity to conflict level, its counter-intuitive results in high and low conflicting situations, etc), other rules have been developed in particular those based on proportional conflict redistribution (PCR) principles.[102,103]

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